

Three papers on  
CHARACTERIZATION AND RECONSTRUCTION  
OF GAUSSIAN PROCESSES  
FROM LEVEL CROSSINGS

by

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This collection contains the following three papers:

- [A] When are the successive zero-crossing intervals of a Gaussian process independent?
- [B] Reconstruction of a stationary Gaussian process from its sign-changes.
- [C] The behaviour of a non-differentiable stationary Gaussian process after a level crossing.



Paper [A]

When are the successive zero-crossing intervals of a Gaussian process independent?





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## 1. INTRODUCTION

This report answers the question: The first of which is whether there is any clustered Gaussian process, a stationary Gaussian process, the process obtained by a unit process. The second is whether the process is a Poisson process, by a unit process. The third is whether the process is a Poisson process, by a unit process.

WHEN ARE

THE SUCCESSIVE ZERO-CROSSING INTERVALS

OF A GAUSSIAN PROCESS INDEPENDENT?

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## Abstract

In this report we will prove that the zero-crossing intervals of a stationary Gaussian process always are dependent. This is true even though there is a Gaussian process whose stream of zero-crossings has the same covariance structure as a Poisson process.





# WHEN ARE THE SUCCESSIVE ZERO-CROSSING INTERVALS OF A GAUSSIAN PROCESS INDEPENDENT ?

by Jacques de Maré

## 1. Introduction

This report answers two questions. The first of which is whether there is any clipped Gaussian process with the same correlation function as the unit process obtained by changing sign at time points given by a Poisson process. By a unit process we will mean a process which just attains the values  $\pm 1$ . A clipped Gaussian process is the unit process which has the same sign as a Gaussian process. An equivalent formulation of the problem is whether the function

$$r(t) = \sin\left(\frac{\pi}{2} e^{-|t|}\right)$$

is a correlation function. This question was stated by McFadden (1958b) and the answer given here is affirmative. McFadden noted that even if such a clipped Gaussian process did exist its zero crossings could not be a Poisson process, since he had shown (1958a) that any Gaussian process with a correlation function having the expansion

$$(1.1) \quad r(t) = 1 - at^2 + b|t|^3 + \dots$$

had dependent zero-crossing intervals. Previously, Palmer (1956) had found that the same is true for all analytical Gaussian processes.

The second question we shall deal with is when the zero-crossing intervals of a stationary Gaussian process are mutually independent. The answer is that the lengths of the successive zero-crossing intervals is a renewal process if and only if the correlation function of the Gaussian process is  $\cos(a \cdot t)$  for some positive constant  $a$ , i.e. the process is a pure cosine-process.

A recent survey article on zero-crossing interval distributions and



other crossing problems is Blake & Lindsey (1973).

## 2. On the Poisson process and the clipped Gaussian process

Let  $X(t)$  be a continuous stationary Gaussian process with zero mean and correlation function  $r(t)$ . Define the clipped Gaussian process  $\xi(t)$  by

$$\xi(t) = \begin{cases} 1 & \text{if } X(t) > 0 \\ -1 & \text{if } X(t) < 0. \end{cases}$$

Then  $\xi(t)$  is a unit process with correlation function  $\frac{2}{\pi} \arcsin r(t)$ . The question to be answered is whether there is a correlation function  $r(t)$  which solves

$$\frac{2}{\pi} \arcsin r(t) = e^{-|t|}.$$

The function  $e^{-|t|}$  is the correlation function of the unit process with sign changes given by a Poisson process with intensity  $1/2$ .

THEOREM The function

$$r(t) = \sin\left(\frac{\pi}{2} e^{-|t|}\right)$$

is a correlation function.

PROOF An even function  $r$  is a correlation function if  $r(0) = 1$  and if it is positive definite. A sufficient condition for positive definiteness is that

$$r(t) = \int_{-\infty}^{\infty} \cos(2\pi\lambda t) f(\lambda) d\lambda$$

where  $f(\lambda) \geq 0$ .

In our case  $r(t) = \sin\left(\frac{\pi}{2} e^{-|t|}\right)$  is absolutely integrable,

$$\int_{-\infty}^{\infty} |r(t)| dt \leq \int_{-\infty}^{\infty} \frac{\pi}{2} e^{-|t|} dt < \infty,$$

so we can define

$$f(\lambda) = \int_{-\infty}^{\infty} \cos(2\pi\lambda t) r(t) dt.$$





To prove the theorem we have to check that  $f$  is integrable and non-negative. We have

$$|f(\lambda)| \leq \int_{-\infty}^{\infty} |r(t)| dt < \infty, \quad \forall \lambda,$$

and further

$$\begin{aligned} \frac{1}{2} f(\lambda/2\pi) &= \frac{1}{\lambda} [\sin(\lambda t) r(t)]_0^{\infty} - \frac{1}{\lambda} \int_0^{\infty} \sin(\lambda t) r'(t) dt = \\ &= -\frac{1}{\lambda^2} [-\cos(\lambda t) r'(t)]_0^{\infty} - \frac{1}{\lambda^2} \int_0^{\infty} \cos(\lambda t) r''(t) dt = \\ &= -\frac{1}{\lambda^2} \int_0^{\infty} \cos(\lambda t) r''(t) dt, \end{aligned}$$

since

$$\begin{aligned} r'(t) &= -\frac{\pi}{2} e^{-t} \cos\left(\frac{\pi}{2} e^{-t}\right), \quad t > 0, \\ r''(t) &= \frac{\pi}{2} e^{-t} \left(\cos\left(\frac{\pi}{2} e^{-t}\right) - \frac{\pi}{2} e^{-t} \sin\left(\frac{\pi}{2} e^{-t}\right)\right), \quad t > 0. \end{aligned}$$

It follows that  $|f(\lambda)| \leq \min(A, B/\lambda^2)$ , where  $A, B$  are finite constants and hence  $f$  is integrable. By the Fourier inversion theorem we get

$$r(t) = \int_{-\infty}^{\infty} \cos(2\pi\lambda t) f(\lambda) d\lambda.$$

It remains to prove that  $f(\lambda)$  is non-negative. We have

$$\begin{aligned} \frac{1}{2} f(\lambda/2\pi) &= \int_0^{\infty} \cos(\lambda t) \sin\left(\frac{\pi}{2} e^{-t}\right) dt = \\ &= \int_0^{\infty} \cos(\lambda t) \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{\pi}{2} e^{-t}\right)^{2k+1}}{(2k+1)!} dt. \end{aligned}$$

By Fubini's theorem we can change the order of integration and summation since

$$\begin{aligned} \sum_{k=0}^{\infty} \int_0^{\infty} \left| \cdot \right| dt &\leq \sum_{k=0}^{\infty} \int_0^{\infty} \frac{\left(\frac{\pi}{2}\right)^{2k+1}}{(2k+1)!} e^{-(2k+1)t} dt \leq \\ &\leq \sum_{k=0}^{\infty} \frac{\left(\frac{\pi}{2}\right)^{2k+1}}{(2k+1)!} = \sinh \frac{\pi}{2} < \infty. \end{aligned}$$

We will then get

$$\frac{1}{2} f(\lambda/2\pi) = \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{\pi}{2}\right)^{2k+1}}{(2k+1)!} \int_0^{\infty} \cos(\lambda t) e^{-(2k+1)t} dt$$

where





$$\int_0^{\infty} \cos(\lambda t) e^{-(2k+1)t} dt = \frac{2k+1}{(2k+1)^2 + \lambda^2},$$

so that

$$\begin{aligned} \frac{1}{2} f(\lambda/2\pi) &= \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{\pi}{2}\right)^{2k+1}}{(2k+1)!} \cdot \frac{2k+1}{(2k+1)^2 + \lambda^2} \\ &= \frac{\pi}{2} \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{\pi}{2}\right)^{2k}}{(2k)!} \cdot \frac{1}{(2k+1)^2 + \lambda^2}. \end{aligned}$$

We now rewrite to obtain an alternating series with decreasing absolute values:

$$\begin{aligned} \frac{1}{2} f(\lambda/2\pi) &= \frac{\pi}{2(1+\lambda^2)} \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{\pi}{2}\right)^{2k}}{(2k)!} \cdot \frac{1+\lambda^2}{(2k+1)^2 + \lambda^2} \\ &= \frac{\pi}{2(1+\lambda^2)} \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{\pi}{2}\right)^{2k}}{(2k)!} \cdot \left\{ 1 - \frac{(2k+1)^2 - 1}{(2k+1)^2 + \lambda^2} \right\} \\ &= \frac{\pi}{2(1+\lambda^2)} \left\{ \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{\pi}{2}\right)^{2k}}{(2k)!} + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\left(\frac{\pi}{2}\right)^{2k}}{(2k)!} \cdot \frac{(2k+1)^2 - 1}{(2k+1)^2 + \lambda^2} \right\} \end{aligned}$$

The first sum equals  $\cos \pi/2 = 0$  and the second sum has alternating terms  $a_k$  with decreasing absolute value:

$$\begin{aligned} \left| \frac{a_k}{a_{k+1}} \right| &= \frac{\left(\frac{\pi}{2}\right)^{2k} \{(2k+1)^2 - 1\}}{(2k)! \{(2k+1)^2 + \lambda^2\}} \cdot \frac{(2k+2)! \{(2k+3)^2 + \lambda^2\}}{\left(\frac{\pi}{2}\right)^{2k+2} \{(2k+3)^2 - 1\}} \\ &= \frac{(2k+2)(2k+1)}{\left(\frac{\pi}{2}\right)^2} \cdot \frac{(2k+1)^2 - 1}{(2k+3)^2 - 1} \cdot \frac{(2k+3)^2 + \lambda^2}{(2k+1)^2 + \lambda^2} \\ &\geq \frac{4 \cdot 3}{\left(\frac{\pi}{2}\right)^2} \cdot \frac{(1 - 1/(2k+1)^2)(2k+1)^2}{(1 - 1/(2k+3)^2)(2k+3)^2} \\ &\geq \frac{48}{\pi^2} \cdot \frac{1-1/9}{1} \cdot \frac{9}{25} \geq 1. \end{aligned}$$

Since the first term is positive we conclude that the total sum is positive, i.e.  $f(\lambda) \geq 0$  for all  $\lambda$  as was to be shown.

**REMARK** There are correlation functions  $\rho(t)$  for unit processes such that  $\sin(\frac{\pi}{2}\rho(t))$  is not positive definite. One example is

$$\rho(t) = \frac{1}{\pi} \{ \arcsin(\cos t) + \arcsin(\cos 3t) \}.$$

The argument is that  $\sin(\frac{\pi}{2}\rho(t))$  coincides with  $\cos 2t$  in a neighbourhood



of zero and if a correlation function is analytical in such a neighbourhood then it has a unique extension to the whole real line. But

$$\sin\left(\frac{\pi}{2}\rho\left(\frac{\pi}{2}\right)\right) = 0 \neq -1 = \cos\left(2 \cdot \frac{\pi}{2}\right).$$

### 3. On the zero-crossing intervals as a renewal process

Let  $\{\xi_n: n = \dots, -2, -1, 0, 1, \dots\}$  be a double sequence of non-negative, independent, identically distributed, stochastic variables with the common distribution  $G$ , not degenerate at zero. Then  $\{\xi_n\}$  is said to be a renewal process. With a Gaussian process having a certain covariance function we will mean the continuous version of the process when it does exist.

THEOREM The successive zero-crossing intervals of a stationary Gaussian process is a renewal process if and only if the Gaussian process has mean zero and correlation function  $\cos(a \cdot t)$  for some positive number  $a$ .

PROOF The "if"-part is trivial since obviously  $G$  is degenerated at  $\pi/a$  if the correlation function is  $\cos(a \cdot t)$ .

Conversely, assume that the successive zero-crossing intervals of the Gaussian process  $X(t)$  is a renewal process. The main step in the proof is then to derive the following functional equation,

$$r(s+t) - r(s)r(t) + r'(s)r'(t) = 0, \quad \forall s > 0, \quad \forall t > 0,$$

where  $r(t)$  is the correlation function of  $X(t)$ . We first note that by symmetry we must have  $EX(t) = 0$ , and since the mean number of renewals during a bounded time interval is finite the correlation function  $r(t)$  is twice differentiable. Without loss of generality we will assume that the covariance function and the correlation function  $r$  coincide and that the time scale is such that

$$r''(0) = -1.$$

Now consider, for any  $s > 0$  and  $t > 0$ , the conditional probabilities





$$P_0(A | B) = P(\text{even number of zero-crossings in } (-s, 0) | \\ \text{odd number in } (0, t) \text{ and } X(0) = 0)$$

$$P_0(A | B^*) = P(\text{even number of zero-crossings in } (-s, 0) | \\ \text{even number in } (0, t) \text{ and } X(0) = 0),$$

where the conditioning on  $X(0) = 0$  is taken in horizontal window sense as Slepian (1962) defines it. Under the renewal assumption it is clear that

$$(3.1) \quad P_0(A | B) = P_0(A | B^*).$$

It is intuitively appealing that these conditional probabilities are the same for all stationary solutions of the stochastic differential equation

$$(3.2) \quad X''(t) + aX'(t) + X(t) = W'(t),$$

( $a \geq 0$ ), where  $W(t)$  is the Wiener process. But these solutions have either degenerated zero-crossing interval distribution or they satisfy (1.1), so they have dependent zero-crossing intervals in any case. We are now going to prove that there are no other solutions of (3.1) than those of (3.2). We will also give a formal proof of McFadden's result that (1.1) implies dependent zero-crossing distances.

To evaluate (3.1) we can either use the representation of the conditioned process invented by Slepian (1962) or do a direct calculation. We give the direct computation here. Use the following notation,

$$D_\epsilon = \frac{X(\epsilon) - X(-\epsilon)}{2\epsilon}$$

$$M_\epsilon = \frac{X(\epsilon) + X(-\epsilon)}{2}.$$

Since there are only finitely many zeroes in  $(-\epsilon, \epsilon)$  and by symmetry we get

$$P_0(A | B) = \lim_{\epsilon \downarrow 0} P(X(-s) < 0 | X(t) < 0, X(-\epsilon) < 0 < X(\epsilon)) = \\ = \lim_{\epsilon \downarrow 0} P(X(-s) < 0 | X(t) < 0, D_\epsilon > 0, |M_\epsilon| < \epsilon D_\epsilon)$$

and

$$P_0(A | B^*) = \lim_{\epsilon \downarrow 0} P(X(-s) < 0 | X(t) > 0, X(-\epsilon) < 0 < X(\epsilon)) = \\ = \lim_{\epsilon \downarrow 0} P(X(-s) < 0 | X(t) > 0, D_\epsilon > 0, |M_\epsilon| < \epsilon D_\epsilon).$$



Now  $(D_\varepsilon, M_\varepsilon, X(-s), X(t))$  is distributed as  $N_4(0, \mathbb{K}(\varepsilon))$  where the covariance matrix  $\mathbb{K}(\varepsilon)$  is

$$\mathbb{K}(\varepsilon) = \begin{pmatrix} \mathbb{K}_{11}(\varepsilon) & \vdots & \mathbb{K}_{12}(\varepsilon) \\ \vdots & \ddots & \vdots \\ \mathbb{K}_{21}(\varepsilon) & \vdots & \mathbb{K}_{22}(\varepsilon) \end{pmatrix}$$

$$\mathbb{K}_{11}(\varepsilon) = \begin{pmatrix} \frac{1-r(2\varepsilon)}{2\varepsilon^2} & 0 \\ 0 & \frac{1+r(2\varepsilon)}{2} \end{pmatrix}$$

$$\mathbb{K}_{12}(\varepsilon) = \mathbb{K}_{21}(\varepsilon) = \begin{pmatrix} \frac{r(s+\varepsilon)-r(s-\varepsilon)}{2\varepsilon} & \frac{r(t-\varepsilon)-r(t+\varepsilon)}{2\varepsilon} \\ \frac{r(s+\varepsilon)+r(s-\varepsilon)}{2} & \frac{r(t-\varepsilon)+r(t+\varepsilon)}{2} \end{pmatrix}$$

$$\mathbb{K}_{22}(\varepsilon) = \begin{pmatrix} 1 & r(s+t) \\ r(s+t) & 1 \end{pmatrix}.$$

Let  $D$  be a random variable having a positive Rayleigh distribution with the density

$$(3.3) \quad f_D(x) = x e^{-x^2/2}, \quad x > 0,$$

and let  $\phi(x, y; \mathbb{K}_{11}(\varepsilon))$  be the bivariate normal density of  $(D_\varepsilon, M_\varepsilon)$ . Then

$$(3.4) \quad (D_\varepsilon \mid |M_\varepsilon| \leq \varepsilon D_\varepsilon) \xrightarrow{L} D, \quad \varepsilon \downarrow 0,$$

since

$$P(D_\varepsilon \leq d \mid |M_\varepsilon| \leq \varepsilon D_\varepsilon) = \frac{\iint_{|y| \leq \varepsilon x \leq \varepsilon d} \phi(x, y; \mathbb{K}_{11}(\varepsilon)) \, dx \, dy}{\iint_{|y| \leq \varepsilon x} \phi(x, y; \mathbb{K}_{11}(\varepsilon)) \, dx \, dy} +$$

$$+ \frac{\iint_{|y| \leq x \leq d} e^{-x^2/2} \, dx \, dy}{\iint_{|y| \leq x} e^{-x^2/2} \, dx \, dy} = \frac{d}{0} \int_0^d x e^{-x^2/2} \, dx, \quad \varepsilon \downarrow 0,$$

by dominated convergence. Further, let  $X_0(-s)$  and  $X_0(t)$  be random variables, jointly distributed with  $D$  such that the conditional distribution of  $X_0(-s), X_0(t)$  given  $D = d > 0$  is bivariate normal:

$$(3.5) \quad (X_0(-s), X_0(t) \mid D = d) \in N_2((r'(s) \cdot d, -r'(t) \cdot d), \begin{pmatrix} R(-s, -s) & R(-s, t) \\ R(t, -s) & R(t, t) \end{pmatrix}),$$

where





$$R(-s, t) = r(s+t) - r(s)r(t) + r'(s)r'(t).$$

By combining (3.3), (3.4), and (3.5) we can now compute the conditional distribution of  $X(-s), X(t)$  given  $D_\epsilon = d, |M_\epsilon| \leq \epsilon D_\epsilon$ . Omitting details, we get

$$(X(-s), X(t) \mid D_\epsilon = d, |M_\epsilon| \leq \epsilon D_\epsilon) \xrightarrow{L} (X_0(-s), X_0(t) \mid D = d), \epsilon \downarrow 0.$$

Here  $D$  corresponds to the variable  $\xi$  in Slepian (1962), p. 104.

If now

$$(3.6) \quad r(t) = a \cos t + b$$

the zero-crossing intervals are dependent. On the other hand, if (3.6) does not hold then there exists  $\delta > 0$  such that the conditional distribution of  $(X_0(-s), X_0(t) \mid D=d)$  is non-singular for all  $s, t; 0 < s < \delta, 0 < t < \delta$ . To see this we consider

$$\dagger = \lim_{\epsilon \downarrow 0} \dagger(\epsilon)$$

and proceed as in Cramér & Leadbetter (1967), p. 203. For any vector  $\theta = (\theta_1, \theta_2, \theta_3, \theta_4)$  we have

$$\theta \dagger \theta^* = \int_{-\infty}^{\infty} |i\lambda\theta_1 + \theta_2 + \theta_3 e^{i\lambda s} + \theta_4 e^{-i\lambda t}|^2 dF(\lambda),$$

where  $F$  is the spectral distribution function of  $r$ . If  $F$  has a continuous component then  $\theta \dagger \theta^*$  is strictly positive for  $\theta \neq 0$ . Otherwise, if  $F$  is purely discrete then  $F$  gives positive mass to two distinct positive points  $\lambda_1$  and  $\lambda_2$  since we have assumed that (3.6) does not hold. But in that case there is a  $\delta > 0$  such that

$$\det \begin{pmatrix} 0 & 1 & \cos \lambda_1 s & \cos \lambda_1 t \\ \lambda_1 & 0 & \sin \lambda_1 s & -\sin \lambda_1 t \\ 0 & 1 & \cos \lambda_2 s & \cos \lambda_2 t \\ \lambda_2 & 0 & \sin \lambda_2 s & -\sin \lambda_2 t \end{pmatrix} > 0$$

for all  $0 < s < \delta, 0 < t < \delta$ , so

$$\sum_{v=1}^2 |i\lambda_v \theta_1 + \theta_2 + \theta_3 e^{i\lambda_v s} + \theta_4 e^{-i\lambda_v t}|^2 > 0$$

for  $\theta \neq 0$ .



We have now proved that if (3.6) does not hold then the distribution in (3.5) is non-singular. We can then proceed and express the probabilities  $P_0(A | B)$  and  $P_0(A | B^*)$  by means of  $X_0(-s)$ ,  $X_0(t)$ , and  $D$  as follows:

$$P_0(A | B) = P(X_0(-s) < 0 | X_0(t) < 0, D > 0),$$

$$P_0(A | B^*) = P(X_0(-s) < 0 | X_0(t) > 0, D > 0).$$

Then (3.1) can be written as

$$P(X_0(-s) < 0 | X_0(t) < 0, D > 0) = P(X_0(-s) < 0 | X_0(t) > 0, D > 0)$$

or

$$\frac{P(X_0(-s) < 0, X_0(t) < 0 | D > 0)}{P(X_0(t) < 0 | D > 0)} = \frac{P(X_0(-s) < 0, X_0(t) > 0 | D > 0)}{P(X_0(t) > 0 | D > 0)}$$

which gives

$$\begin{aligned} P(X_0(t) < 0 | D > 0) \cdot \{P(X_0(-s) < 0, X_0(t) > 0 | D > 0) + \\ + P(X_0(-s) < 0, X_0(t) < 0 | D > 0)\} = \\ = P(X_0(-s) < 0, X_0(t) < 0 | D > 0). \end{aligned}$$

Finally we get

$$\begin{aligned} P(X_0(-s) < 0 | D > 0) P(X_0(t) < 0 | D > 0) - \\ - P(X_0(-s) < 0, X_0(t) < 0 | D > 0) = 0. \end{aligned}$$

The lefthand side of this equation equals

$$\begin{aligned} \int_0^\infty \{P(X_0(-s) < 0 | D = x) P(X_0(t) < 0 | D = x) - \\ - P(X_0(-s) < 0, X_0(t) < 0 | D = x)\} f_D(x) dx \end{aligned}$$

where the integrand can be expressed as

$$I = f_D(x) \int_0^\rho \frac{1}{2\pi(1-y)^2} \exp\left(-\frac{1}{2(1-y)^2} Q(x, y)\right) dy$$

with

$$Q(x, y) = \frac{x^2 r'(-s)^2}{R(-s, -s)} + \frac{x^2 r'(t)^2}{R(t, t)} - \frac{2x^2 y r'(-s) r'(t)}{\sqrt{R(-s, -s)R(t, t)}}$$

and

$$\rho = R(-s, t) / \sqrt{R(-s, -s)R(t, t)},$$





by e.g. Cramér & Leadbetter (1967), p. 27. Hence

$$I > 0 \quad \text{if } R(-s, t) > 0,$$

$$I < 0 \quad \text{if } R(-s, t) < 0,$$

so .

$$P_0(A | B) = P_0(A | B^*) \quad \text{if and only if } R(-s, t) = 0.$$

But  $R(-s, t) = 0$  for  $0 < s < \delta, 0 < t < \delta$  is the same as

$$r(s+t) - r(s)r(t) = -r'(s)r'(t) \quad \text{for } 0 < s < \delta, 0 < t < \delta.$$

Since the lefthand side is twice differentiable so is the right,

$$r''(s+t) - r''(s)r(t) = -r'''(s)r'(t) \quad \text{for } 0 < s < \delta, 0 < t < \delta,$$

and we get, as  $s \downarrow 0$ ,

$$r''(t) + r'''(+0)r'(t) + r(t) = 0 \quad \text{for } 0 < t < \delta.$$

Now, since  $-r''(t)$  is a correlation function,  $r'''(+0) \geq 0$ . If  $r'''(+0) = 0$  we get

$$r''(t) + r(t) = 0 \Rightarrow r(t) = \cos t$$

so the process is of the pure cosine-type with completely dependent zero-crossing distances. If, on the other hand,  $r'''(+0) > 0$  we get

$$(3.7) \quad r(t) = 1 - \frac{t^2}{2} + \frac{r'''(+0)}{3!} |t|^3 - \dots$$

But this expansion is the same as (1.1) and then  $X$  has dependent zero-crossing distances by McFadden (1958a).

We will conclude this section with a formal proof of the dependence based on the idea indicated by McFadden. Let  $r$  have the expansion (3.7) with  $r'''(+0) > 0$  and use the following notations,

$$N_x = \text{the number of downcrossings in } (0, x)$$

$$U_x = \text{the number of upcrossings in } (0, x).$$

Then there are integrable functions  $f_1$  and  $f_2$  such that the conditional expectations of  $N_x$  and  $U_x$  given a zero upcrossing at 0 are given by



$$E(N_x \mid \text{zero upcrossing at } 0) = \int_0^x f_1(t) dt, \quad x > 0,$$

$$E(U_x \mid \text{zero upcrossing at } 0) = \int_0^x f_2(t) dt, \quad x > 0.$$

If we write

$$c_t = \left\{ -r''(t) - \frac{r(t)r'(t)^2}{1-r(t)^2} \right\} / \left\{ -r''(0) - \frac{r'(0)^2}{1-r(0)^2} \right\}$$

it can be shown that

$$\frac{f_2(t)}{f_1(t)} = \frac{\sqrt{1-c_t^2} + c_t \arccos(-c_t)}{\sqrt{1-c_t^2} - c_t \arccos(c_t)}, \quad t > 0,$$

(cf. Longuet-Higgins (1962)). As is easily seen, if (3.7) holds, then

$$c_t \rightarrow -\frac{1}{2}, \quad t \downarrow 0$$

and hence

$$(3.8) \quad \frac{f_2(t)}{f_1(t)} \rightarrow \frac{\sqrt{3}/2 - 1/2 \cdot \pi/3}{\sqrt{3}/2 + 1/2 \cdot 2\pi/3} = A > 0, \quad t \downarrow 0.$$

Now, under the renewal assumption the zero-crossing interval distribution  $G$  is absolutely continuous with respect to the measure on  $[0, \infty)$  induced by  $E(N_x \mid \text{zero upcrossing at } 0)$ , which has a density  $f_1$ . Hence we find that  $G$  will have a density  $g$ , and the following equality holds:

$$f_2 = g * f_1,$$

(where  $*$  denotes convolution). It follows from (3.8) that

$$\frac{g * f_1}{f_1}(t) \rightarrow A > 0, \quad t \downarrow 0,$$

so there is a  $\delta > 0$  such that

$$g * f_1(t) > \frac{A}{2} f_1(t) \quad \text{if } 0 < t < \delta.$$

Hence

$$(3.9) \quad \frac{\int_0^x g * f_1(t) dt}{\int_0^x f_1(t) dt} > \frac{\int_0^x \frac{A}{2} f_1(t) dt}{\int_0^x f_1(t) dt} = \frac{A}{2} > 0 \quad \text{if } 0 < x < \delta.$$

On the other hand





$$\begin{aligned}
 \frac{\int_0^x g * f_1(t) dt}{\int_0^x f_1(t) dt} &= \frac{\iint_{s+t \leq x} g(s)f_1(t) ds dt}{\int_0^x f_1(t) dt} \\
 &\leq \frac{\int_{s=0}^x \int_{t=0}^x g(s)f_1(t) ds dt}{\int_0^x f_1(t) dt} \\
 &= \int_0^x g(s) ds \rightarrow 0, x \rightarrow 0
 \end{aligned}$$

which contradicts (3.9).

To summarize, we have found that even those processes which satisfy (3.1) have dependent zero-crossing intervals and the proof of the theorem is concluded.

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Paper [B]

Reconstruction of a stationary Gaussian process from its sign-changes.





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RECONSTRUCTION OF A STATIONARY GAUSSIAN  
PROCESS FROM ITS SIGN-CHANGES

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Abstract

The sign-changes over the entire real line of a Gaussian process with unit variance and mean zero are observed. The Gaussian process is reconstructed by an interpolator which is optimal in the class of linear interpolators based on that process which is one when the original process is positive and minus one when it is negative. Sufficient conditions for the interpolator to be a convolution of the sign process and a square integrable function are given. Analytical expressions of the interpolation error are derived, and the behaviour of the interpolator is studied by means of simulations.

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RECONSTRUCTION OF A STATIONARY GAUSSIAN  
PROCESS FROM ITS SIGN-CHANGES

by Jacques de Maré

0. Introduction

In this paper we will discuss the problem of recovering an ergodic Gaussian process with unit variance and mean zero between its zero-crossings by means of observations of all its sign-changes over the entire real line. We assume that the correlation function is known which is actually no restriction, since in this case it is always possible to estimate it perfectly.

Bar-David (1975) pointed out that generally it is impossible to make the interpolation without error. The best quadratic mean interpolator is the conditional expectation of the process conditioned on its sign changes. Unfortunately, it seems difficult to obtain an explicit evaluation of this expression and therefore we will instead consider the best linear interpolator of the process from the clipped process, i.e. the process which is 1 when the original process is positive and -1 when it is negative.

Sufficient conditions for the best linear interpolator to have a simple form as a quadratic mean integral of a deterministic function with random sign-changes are given. Since the interpretation of quadratic mean integration when one has just one realization is dubious the question of convergence with probability one is also discussed.

The analytic expression of the interpolation error is given together with some numerical examples. In one section we illustrate the use of the best linear interpolator by some simulation studies where the interpolator is plotted together with the original process. In an appendix we



will describe the principles of the algorithm used to obtain simulations of a process with a given spectral density.

### 1. The main results

Let  $X$  be a continuous stationary Gaussian process having zero mean, unit variance and an integrable correlation function  $r$  with spectral density  $f$ . Define the clipped process  $\xi$  by the equation

$$\xi(t) = \begin{cases} 1 & \text{if } X(t) > 0 \\ 0 & \text{if } X(t) = 0 \\ -1 & \text{if } X(t) < 0 \end{cases}.$$

The problem here is to reconstruct  $\{X(t): t \in \mathbb{R}\}$  from an observation of  $\{\xi(t): t \in \mathbb{R}\}$ . We will assume that  $r$  is known but this is no real restriction since  $X$  is ergodic which makes it possible to estimate  $r$  without error from our observation (see Hinich (1967)). The best reconstruction of  $X$  in quadratic mean sense is

$$\tilde{X}(t) = E[X(t) \mid \{\xi(s): s \in \mathbb{R}\}].$$

This conditional expectation seems difficult to evaluate explicitly so we turn to the problem of evaluating the best reconstruction in the more restricted class of linear interpolators. Since  $\xi$  itself is a stationary process which is stationarily correlated with  $X$  we can use the Kolmogorov-Wiener second order theory to obtain the best linear interpolator  $\hat{X}$  based on  $\xi$ . In fact, as will be shown, we can write

$$(1.1) \quad \xi = \sqrt{2/\pi} X + Y,$$

where  $\text{Cov}[X(s), Y(t)] = 0 \quad \forall s \in \mathbb{R} \quad \forall t \in \mathbb{R}$ . Hence in the second order linear theory the problem of interpolating  $X$  from an observation of  $\xi$  is the





same as the problem of reconstructing  $X$  when it is observed together with orthogonal noise. The general solution of this problem is

$$(1.2) \quad \hat{X}(t) = \int_{-\infty}^{\infty} e^{i\lambda t} g(\lambda) dZ(\lambda) \quad (\text{q.m.}),$$

where  $Z$  is the spectral process of  $\xi$  and where  $g$  is proportional to the ratio of the spectral densities  $f$  and  $\varphi$  of  $X$  and  $\xi$ , respectively. However in the problem of interpolating  $X$  from the clipped process  $\xi$ , we observe  $\xi$  directly and not the spectral process  $Z$ . Hence we want a more direct representation of  $\hat{X}$  in terms of  $\xi$  than via the spectral process  $Z$ , so we are going to give sufficient conditions for

$$(1.3) \quad \hat{X}(t) = \int_{-\infty}^{\infty} \xi(t-u) q(u) du,$$

where  $q$  is a square integrable covariance function, namely

$$(1.4) \quad q(u) = 2^{-1/2} \pi^{-3/2} \int_{-\infty}^{\infty} e^{i\lambda u} f(\lambda) / \varphi(\lambda) d\lambda.$$

The main step in the proof is to show that  $q$  is well defined. Note that the stochastic integral in (1.3) is of a simple kind since  $q$  is a deterministic function given once and for all for any specific covariance structure, while  $\xi$  is a stochastic process which just attains the values  $-1, 0$  and  $1$ .

To justify (1.1) the cross-correlation function between  $\xi$  and  $X$  has to be computed.

LEMMA 1.1 The cross-correlation function between  $\xi$  and  $X$  is given by

$$(1.5) \quad \text{Cov}[\xi(s), X(t)] = \sqrt{2/\pi} r(s-t),$$

which implies the relation (1.1) if  $Y$  is defined by  $Y(t) = \xi(t) - \sqrt{2/\pi} X(t)$ .



PROOF  $\text{Cov}[\xi(s), X(t)] = E[\xi(s) \cdot X(t)] = E[X(t) \mid X(s) > 0] P[X(s) > 0] -$   
 $- E[X(t) \mid X(s) < 0] P[X(s) < 0] = E[X(t) \mid X(s) > 0] =$   
 $= \int_0^{\infty} x r(s-t) \sqrt{2/\pi} e^{-x^2/2} dx = \sqrt{2/\pi} r(s-t). \quad \square$

Note that for non-Gaussian processes it is not necessary that the regression of  $X(t)$  on  $X(s)$  is linear to obtain  $\text{Cov}[\xi(s), X(t)] = \text{constant} \cdot r(s-t)$ . On the contrary it is sufficient that

$$E[X(t) \mid X(s) = x] = G(x) \cdot r(s-t).$$

To derive (1.3) we need the following characterization of the best linear interpolator  $X^*$  based on  $\xi$ ,

CHARACTERIZATION 1.2 A linear interpolator  $X^*$  of  $X$  from the  $\xi$ -process is the best linear interpolator if and only if

$$(1.6) \quad \text{Cov}[\xi(s), X^*(t)] = \sqrt{2/\pi} \cdot r(s-t) \quad \forall s \in R \quad \forall t \in R.$$

This characterization is valid since a linear interpolator  $X^*$  is best linear if and only if the interpolation error  $X(t) - X^*(t)$  is orthogonal to the  $\xi$ -process for all  $t$ . But

$$E\{[X(t) - X^*(t)] \cdot \xi(s)\} = \sqrt{2/\pi} \cdot r(s-t) - \text{Cov}[\xi(s), X^*(t)],$$

which gives (1.6).

We are now going to consider linear interpolators which are convolutions of the  $\xi$ -process and a square integrable function as in (1.3).

LEMMA 1.3 If  $h$  is a square integrable function then  $X^*$  given by

$$(1.7) \quad X^*(t) = \int_{-\infty}^{\infty} \xi(t-u) h(u) du,$$

is well defined in quadratic mean.





PROOF Let the correlation function of the  $\xi$ -process be  $\rho$ . Then

$$\begin{aligned}\text{Var}[X^*(t)] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E[\xi(t-u)\xi(t-v)] h(u) h(v) du dv \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho(u-v) h(u) h(v) du dv.\end{aligned}$$

But

$$\begin{aligned}\rho(u-v) &= E[\xi(u)\xi(v)] = P[X(u)X(v)>0] - P[X(u)X(v)<0] = \\ &= \frac{2}{\pi} \cdot \arcsin[r(u-v)].\end{aligned}$$

Now we have assumed that  $r$  is integrable and hence so is  $\rho$ , which gives that  $\int_{-\infty}^{\infty} \rho(t-v) h(v) dv$  is square integrable as a function of  $t$ , since the convolution between an integrable function and a square integrable function is square integrable. We get

$$\text{Var}[X^*(t)] = \int_{-\infty}^{\infty} h(u) (\rho * h)(u) du < \infty. \quad \square$$

Note that if  $X$  had any other mean level than zero we would not have got a simple expression for  $\rho$ .

We can now express the characterization (1.6) in terms of the weight function  $h$  in (1.7). Since it follows from (1.6) that  $X^*$  of (1.7) is the best linear interpolator if

$$\begin{aligned}\sqrt{\frac{2}{\pi}} r(s-t) &= \text{Cov}[\xi(s), X^*(t)] = \int_{-\infty}^{\infty} E[\xi(s) \cdot \xi(t-u)] h(u) du = \\ &= \int_{-\infty}^{\infty} \rho(t-s-u) h(u) du = \rho * h(t-s),\end{aligned}$$

we find that the linear interpolator  $X^*$  of (1.7) is best linear if  $h$  is a square integrable solution of the integral equation

$$(1.8) \quad \rho * h = \sqrt{2/\pi} r.$$



The solution  $h$  is most easily obtained by applying the inverse Fourier transform on (1.8) as in the following theorem which gives sufficient conditions for the representation (1.3) in quadratic mean. Remember that the integrability of  $r$  implies that of  $\rho$  as in the proof of Lemma 1.3, and hence the  $\xi$ -process has a spectral density  $\varphi$  if  $\int_{-\infty}^{\infty} |r| < \infty$ .

THEOREM 1.4 Let  $X$  be a stationary Gaussian process with mean zero, unit variance and an absolute integrable covariance function  $r$ . If  $X$  has finite third absolute spectral moment then the best linear interpolator  $\hat{X}$  of  $X$  from its clipped process  $\xi$  is

$$(1.9) \quad \hat{X}(t) = \int_{-\infty}^{\infty} \xi(t-u) q(u) du \quad (\text{q.m.}),$$

where the impulse response function

$$(1.10) \quad q(u) = 2^{-1/2} \pi^{-3/2} \int_{-\infty}^{\infty} e^{i\lambda t} f(\lambda) / \varphi(\lambda) d\lambda,$$

is square integrable and  $f$  and  $\varphi$  are spectral densities of  $X$  and  $\xi$  respectively.

The main step in the proof is given by the following lemma, the proof of which is given in the next section.

LEMMA 1.5 Under the hypothesis of Theorem 1.4

$$\int_{-\infty}^{\infty} f(\lambda) / \varphi(\lambda) d\lambda < \infty.$$

Proof of Theorem 1.4 Set

$$g(\lambda) = \begin{cases} 2^{-1/2} \pi^{-3/2} \frac{f}{\varphi}(\lambda) & \text{if } \varphi(\lambda) > 0 \\ 0 & \text{if } \varphi(\lambda) = 0 \end{cases}.$$

Now by (1.1) we have

$$\varphi \geq \frac{2}{\pi} f,$$



and hence  $g$  is a bounded function. Also, by Lemma 1.5,  $g$  is integrable and hence square integrable which implies that its Fourier transform

$$q(u) = \int_{-\infty}^{\infty} e^{i\lambda u} g(\lambda) d\lambda,$$

is square integrable as well.

Since

$$\rho = \frac{2}{\pi} \arcsin r,$$

is integrable it follows that  $\rho * q$  is square integrable. The inverse Fourier transform of  $\rho * q$  is  $2\pi \varphi \cdot g = \sqrt{\frac{2}{\pi}} f$  which gives

$$(1.11) \quad \rho * q = \sqrt{\frac{2}{\pi}} r.$$

Hence we have proved (1.8) which establishes the theorem.  $\square$

In applications, it is often easier to interpret convergence with probability one than convergence in quadratic mean. Therefore it is interesting to discuss when the integral (1.9) converges almost surely. Some sufficient conditions are easily obtainable.

COROLLARY 1.6 If  $\int |q| < \infty$  where  $q$  is defined by (1.10) then

$$\hat{X}(t) = \int_{-\infty}^{\infty} \xi(t-u) q(u) du \quad \text{a.s.}$$

PROOF The result is immediate since

$$\int_{-\infty}^{\infty} |\xi(t-u) q(u)| du = \int_{-\infty}^{\infty} |q(u)| du < \infty, \quad \square$$

COROLLARY 1.7 If  $r \geq 0$  and  $q \geq 0$  then

$$\hat{X}(t) = \int_{-\infty}^{\infty} \xi(t-u) q(u) du \quad \text{a.s.}$$

for  $q$  given by (1.10).





PROOF We have by (1.11)

$$\sqrt{\frac{2}{\pi}} r = \rho * q$$

But now  $r$  is integrable and hence

$$\begin{aligned} \infty > \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} |r(s)| ds &= \int_{-\infty}^{\infty} |\rho * q(s)| ds = \\ &= \int_{s=-\infty}^{\infty} \left| \int_{t=-\infty}^{\infty} \rho(s-t) q(t) dt \right| ds. \end{aligned}$$

But  $\rho \geq 0$  and  $q \geq 0$  so by Fubini's theorem

$$\begin{aligned} \infty > \int \int_{\mathbb{R}^2} \rho(s-t) q(t) ds dt &= \int \int_{\mathbb{R}^2} \rho(s) q(t) ds dt = \\ &= \int_{-\infty}^{\infty} \rho(s) ds \int_{-\infty}^{\infty} q(t) dt \end{aligned}$$

and hence  $\int |q| < \infty$ . Now the result follows by Corollary 1.6.  $\square$

REMARK 1.8 If the assumption of the integrability of  $r$  is weakened to square integrability, then the proof of Lemma 1.5 does not work. This can be remedied e.g. by assuming compact support of  $f$ . Then  $g$  is bounded and integrable which is enough to ensure that  $\rho * q$  and  $r * q$  are square integrable. We get

$$\begin{aligned} \text{Var}[\hat{X}(t)] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho(u-v) q(u) q(v) du dv = \\ &= \int_{-\infty}^{\infty} \rho * q(u) q(u) du < \infty. \end{aligned}$$

Still  $\rho * q = \sqrt{2/\pi} \cdot r$  and hence the conclusion of Theorem 1.4 holds under this assumption but the proof of Corollary 1.7 fails.

An intuitive reason which explains why the convolution

$$\int_{-\infty}^{\infty} \xi(t-u) q(u) du,$$



should converge almost surely is that the integrand is a square integrable function with random sign changes and since the harmonic series with random signs converges so should the integral. Theorem 1.11 makes this idea precise, but first we give a definition of uniform mixing.

DEFINITION 1.9 A stationary process  $X$  is said to satisfy the uniform mixing condition if

$$\sup_{A \in F_t, B \in G_{t+u}} |P(A|B) - P(A)| \rightarrow 0, \quad u \rightarrow \infty,$$

where  $F_t$  is the  $\sigma$ -algebra generated by  $\{X(s): s \leq t\}$  and  $G_{t+u}$  the  $\sigma$ -algebra generated by  $\{X(s): s \geq t+u\}$ .

REMARK 1.10 A stationary Gaussian process  $X$  with covariance function  $r$  is uniformly mixing if and only if  $r$  has compact support, (cf. Ibragimov-Linnik (1971), p. 312).

THEOREM 1.11 If the Gaussian process in Theorem 1.4 is uniformly mixing then

$$\hat{X}(t) = \int_{-\infty}^{\infty} \xi(t-u) q(u) du \quad \text{a.s.}$$

REMARK 1.12 There are covariance functions with compact supports which satisfy the hypothesis of Theorem 1.4. One example is

$$r(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda t} \left( \frac{\sin \lambda}{\lambda} \right)^{2n} d\lambda \quad \text{for } n \geq 3.$$

PROOF (Of Theorem 1.11) it is enough to prove that

$$\eta = \int_0^{\infty} \xi(s) q(s) ds,$$

converges with probability one.





Define  $F_t$  as in Definition 1.9 and note that  $\eta$  is well defined as a quadratic mean limit, and hence we can set

$$\eta_t = E(\eta | F_t) = \int_0^\infty E(\xi(s) | F_t) q(s) ds.$$

The right hand-side is a version of  $\eta_t$  and there is no problem with measurability since a regular conditional probability exists. By Remark 1.12,  $r$  has compact support so there is an integer  $k$  with

$$\begin{aligned} \eta_t &= \int_0^{t+k} E(\xi(s) | F_t) q(s) ds = \\ &= \int_0^t \xi(s) q(s) ds + \int_t^{t+k} E(\xi(s) | F_t) q(s) ds. \end{aligned}$$

The relation (1.10) states that  $q$  is a covariance function and therefore it is uniformly continuous. Since it is also square integrable it follows that  $q(s) \rightarrow 0$ ,  $s \rightarrow \infty$ , which together with  $|E[\xi(s) | F_t]| \leq 1$  implies that

$$\eta_t - \int_0^t \xi(s) q(s) ds = \int_t^{t+k} E(\xi(s) | F_t) q(s) ds \rightarrow 0, \quad t \rightarrow \infty.$$

Now  $\eta_t$ ,  $t \geq 0$  is a convergent martingale and hence

$$\int_0^\infty \xi(s) q(s) ds,$$

converges a.s. □

## 2. The proof of the crucial lemma

In this section we are going to prove that

$$q(u) = 2^{-1/2} \pi^{-3/2} \int_{-\infty}^\infty e^{i\lambda u} f(\lambda) / \varphi(\lambda) d\lambda,$$

of (1.10) is well defined which is the hard point in the proof of Theorem 1.4. We begin by giving an expansion of  $\varphi$  in terms of  $f$ .



LEMMA 2.1 Let  $r$  be an integrable correlation function with spectral density  $f$ . Then the spectral density  $\varphi$  of  $\rho = (2/\pi)\arcsin(r)$  satisfies

$$(2.1) \quad \varphi = \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!} (2n+1)^{-1} f^{(2n+1)*}.$$

PROOF Since  $r$  is integrable so is  $\rho$  and hence

$$\varphi(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda t} \frac{2}{\pi} \arcsin r(t) dt.$$

Now  $\arcsin r(t) = \sum_{k=0}^{\infty} a_k (r(t))^k$  for almost all  $t$  where  $a_k$  are the same coefficients as in the series (2.1), so we can write

$$\varphi(\lambda) = \pi^{-2} \int_{-\infty}^{\infty} e^{i\lambda t} \sum_{k=0}^{\infty} a_k (r(t))^k dt.$$

This expression is absolutely convergent since

$$\int_{-\infty}^{\infty} \sum_{k=0}^{\infty} a_k |r(t)|^k dt = \int_{-\infty}^{\infty} \arcsin |r(t)| dt < \infty$$

and hence by the Fubini theorem

$$\begin{aligned} \varphi(\lambda) &= \pi^{-2} \sum_{k=0}^{\infty} a_k \int_{-\infty}^{\infty} e^{i\lambda t} (r(t))^k dt \\ &= \pi^{-2} \sum_{k=0}^{\infty} a_k 2\pi f^{k*}(\lambda) \\ &= \frac{2}{\pi} \sum_{k=0}^{\infty} a_k f^{k*}(\lambda). \end{aligned}$$

□

In order to show that  $f/\varphi$  is integrable we want to estimate  $\varphi$  from below. The following local form of the central limit theorem is due to Petrov (1965) and it gives a useful approximation of the convolutions in the series (2.1).



LEMMA 2.2 (Petrov). If  $f$  is a bounded probability density with mean zero, variance one and finite third absolute moment then for all sufficiently large  $n$

$$(2.2) \quad \left| f^{n*}(\lambda) - \frac{1}{\sqrt{n}} \phi\left(\frac{\lambda}{\sqrt{n}}\right) \right| \leq \frac{C}{n\left(1 + \frac{|\lambda|^3}{\sqrt{n}^3}\right)},$$

where  $\phi$  denotes the standard normal density.

We restate Lemma 1.5 as follows,

LEMMA 2.3 Let  $r$  be an integrable correlation function and set

$$\begin{cases} f(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda t} r(t) dt \\ \varphi(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda t} \frac{2}{\pi} \arcsin r(t) dt. \end{cases}$$

If

$$(2.3) \quad \int_{-\infty}^{\infty} |\lambda|^3 f(\lambda) d\lambda < \infty$$

and

$$g(\lambda) = \begin{cases} 2^{-1/2} \pi^{-3/2} f(\lambda)/\varphi(\lambda) & \text{if } \varphi(\lambda) > 0, \\ 0 & \text{if } \varphi(\lambda) = 0 \end{cases},$$

then

$$\int_{-\infty}^{\infty} g(\lambda) d\lambda < \infty.$$

PROOF Since (2.3) implies that

$$\lambda_2 = \int_{-\infty}^{\infty} \lambda^2 f(\lambda) d\lambda < \infty,$$

we can without loss of generality assume that

$$\lambda_2 = 1.$$





We note that it is enough to prove that

$$(2.4) \quad \exists \varepsilon > 0 \exists \lambda' \geq 1 \text{ with } |\lambda| \geq \lambda' \Rightarrow \varphi(\lambda) > \varepsilon/\lambda^2,$$

since then by (1.1)

$$0 \leq g(\lambda) = 2^{-1/2} \pi^{-3/2} \frac{f(\lambda)}{\varphi(\lambda)} \leq 2^{-3/2} \pi^{-1/2} I_{[-\lambda', \lambda']}(\lambda) + \frac{\lambda^2 f(\lambda)}{\varepsilon},$$

where the right hand side is integrable. ( $I_A$  denotes the indicator function of  $A$ .)

To prove (2.4) we use the expansion in Lemma 2.1, which yields

$$\begin{aligned} \varphi(\lambda) &= \frac{2}{\pi} \sum_{v=0}^{\infty} \frac{(2v-1)!!}{(2v)!!(2v+1)} f^{(2v+1)*}(\lambda) \geq \\ &\geq \frac{2}{\pi} \sum_{v=1}^{\infty} \frac{1}{\sqrt{\pi}} \frac{1}{e} \frac{1}{\sqrt{v}} \frac{1}{2v+1} f^{(2v+1)*}(\lambda), \end{aligned}$$

since

$$\frac{(2v+1)!!}{(2v)!!(2v+1)} \geq \frac{1}{e\sqrt{\pi} \sqrt{v} (2v+1)},$$

by Stirling's formula.

Since  $r$  is integrable, the density  $f$  is bounded with

$$\int |\lambda|^3 f(\lambda) d\lambda < \infty,$$

and then (2.2) holds for sufficiently large  $n$ . Hence

$$\begin{aligned} \varphi(\lambda) &\geq \frac{2}{\pi} \sum_{v=n_0}^{\infty} \frac{1}{\sqrt{\pi}} \frac{1}{e} \frac{1}{\sqrt{v}} \frac{1}{(2v+1)} \left( \frac{1}{\sqrt{2v+1}} \vartheta\left(\frac{\lambda}{\sqrt{2v+1}}\right) - \frac{C}{(2v+1) \left(1 + \frac{|\lambda|^3}{(2v+1)^{3/2}}\right)} \right) \\ &\geq \frac{2}{\pi} \frac{1}{e\sqrt{\pi}} \sum_{v=n_0}^{\infty} \frac{1}{(2v+1)^2} \vartheta\left(\frac{\lambda}{\sqrt{2v+1}}\right) - \frac{2}{\pi} \frac{1}{e\sqrt{\pi}} C \sum_{v=n_0}^{\infty} \frac{v^{-1}}{(v^{3/2} + |\lambda|^3)} = \\ &= \frac{2}{\pi} \frac{1}{e\sqrt{\pi}} \Sigma_1 - \frac{2}{\pi} \frac{C}{e\sqrt{\pi}} \Sigma_2, \text{ say.} \end{aligned}$$



Put

$$g_{\lambda}(x) = \frac{1}{(2x+1)^2} \varphi\left(\frac{\lambda}{\sqrt{2x+1}}\right), \quad |\lambda| \geq 1, \quad x \geq 1$$

and

$$h_{\lambda}(x) = \frac{1}{x(x^{3/2} + |\lambda|^3)}.$$

Then  $\Sigma_1 \geq \int_{n_0}^{\infty} g_{\lambda}(x) dx = \sup_{x \geq n_0} g_{\lambda}(x)$  since  $g_{\lambda}$  is unimodal.

Now  $\sup_{x \geq n_0} g_{\lambda}(x) \leq \frac{2}{\lambda^4}$  and

$$\begin{aligned} \int_{n_0}^{\infty} g_{\lambda}(x) dx &= \frac{1}{2} \int_{2n_0+1}^{\infty} y^{-2} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\lambda^2}{2y}\right) dy \\ &= \frac{1}{2\sqrt{2\pi}} \int_0^{(2n_0+1)^{-1}} \exp(-\lambda^2 z/2) dz \\ &= \frac{1}{2\sqrt{2\pi}} [-2/\lambda^2 \exp(-\lambda^2 z/2)]_0^{(2n_0+1)^{-1}} \\ &\geq C_{n_0} \cdot \lambda^{-2} > 0. \end{aligned}$$

Hence  $\Sigma_1 \geq C'_{n_0} \cdot \lambda^{-2} > 0$  for all  $\lambda \geq \lambda''$ .

Further

$$\Sigma_2 \leq \int_{n_0-1}^{\infty} h_{\lambda}(x) dx = \int_{n_0-1}^{\infty} \frac{dx}{x(x^{3/2} + |\lambda|^3)}.$$

Now  $\lambda^{2.5} h_{\lambda}(x) \rightarrow 0$ ,  $\lambda \rightarrow \infty$  and hence  $\Sigma_2 = o(\lambda^{-2.5})$ ,  $\lambda \rightarrow \infty$ , by monotone convergence. We have obtained

$$\varphi(\lambda) \geq \frac{2}{\pi} \frac{1}{e\sqrt{\pi}} C'_{n_0} / \frac{1}{\lambda^2} - o(\lambda^{-2.5}) \geq \frac{C''_{n_0}}{\lambda^2} \text{ for all } \lambda > \lambda''',$$

which, as was pointed out above, is enough to establish the integrability of  $g$ . □



### 3. The behaviour of the interpolator in three examples

From simulations it is possible to get some idea of the behaviour of the best linear interpolator. Here we will present some realizations of the best linear interpolator together with those of the Gaussian processes which the interpolator is trying to reconstruct. Gaussian processes with three different spectral densities are simulated and the following notations are used. The Gaussian process  $X$  with mean zero and unit variance has the correlation function  $r$  and spectral density  $f$ . The clipped process  $\xi = \text{sign } X$  has the correlation function  $\rho = (2/\pi) \cdot \arcsin(r)$  and spectral density  $\varphi$ . The best linear interpolator  $\hat{X}$  is calculated from  $\xi$  as

$$\hat{X}(t) = \int_{-\infty}^{\infty} \xi(t-u) q(u) du,$$

according to Theorem 1.4 and Remark 1.8. The impulse response function  $q$  is given by

$$q(u) = \int_{-\infty}^{\infty} e^{i\lambda u} g(\lambda) d\lambda,$$

with  $g(\lambda) = 2^{-1/2} \pi^{-3/2} f(\lambda)/\varphi(\lambda)$  if  $\varphi(\lambda) \neq 0$  and 0 otherwise.

For technical information on the simulations the reader is referred to the appendix.

EXAMPLE 3.1 Let  $X$  be Gaussian low frequency white noise with spectral density  $f = \frac{1}{2} I_{[-1,1]}$  as in Figure 3.3. The corresponding correlation function is  $r(t) = \sin(t)/t$  (Figure 3.1), while the clipped process has the nondifferentiable correlation function  $\rho$  with

$$\rho(t) = (2/\pi) \arcsin r(t) = (2/\pi) \arcsin \left( \frac{\sin t}{t} \right),$$

(Figure 3.2). The spectral density  $\varphi$  of  $\rho$  is more outspread than  $f$  as one expects since no spectral moments exist (Figure 3.3). The ratio





$g = 2^{-1/2} \pi^{-3/2} f/\varphi$  is plotted in Figure 3.4 while its Fourier transform, the impulse response function  $q$ , is given in Figure 3.5. A simulation of  $X$  and the corresponding realization of  $\xi$  are found in Figures 3.6 and 3.7, while the interpolator  $\hat{X}$  is plotted in Figure 3.8.

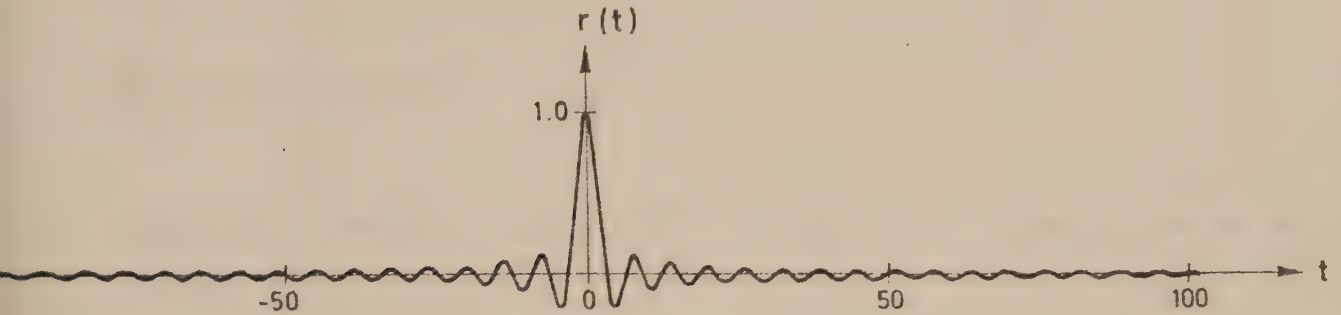


FIGURE 3.1 The correlation function  $r(t) = \sin(t)/t$

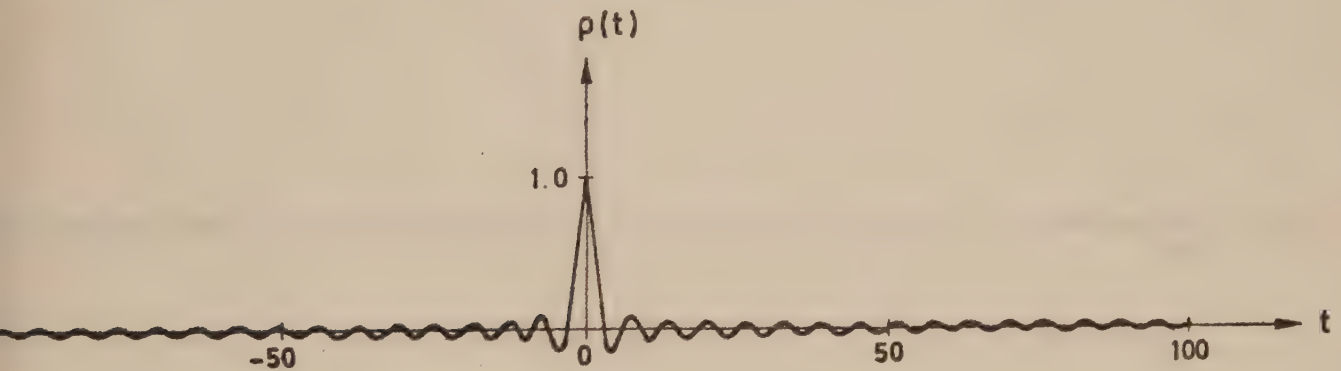


FIGURE 3.2 The correlation function  $\rho(t) = \frac{2}{\pi} \arcsin \left( \frac{\sin t}{t} \right)$ .

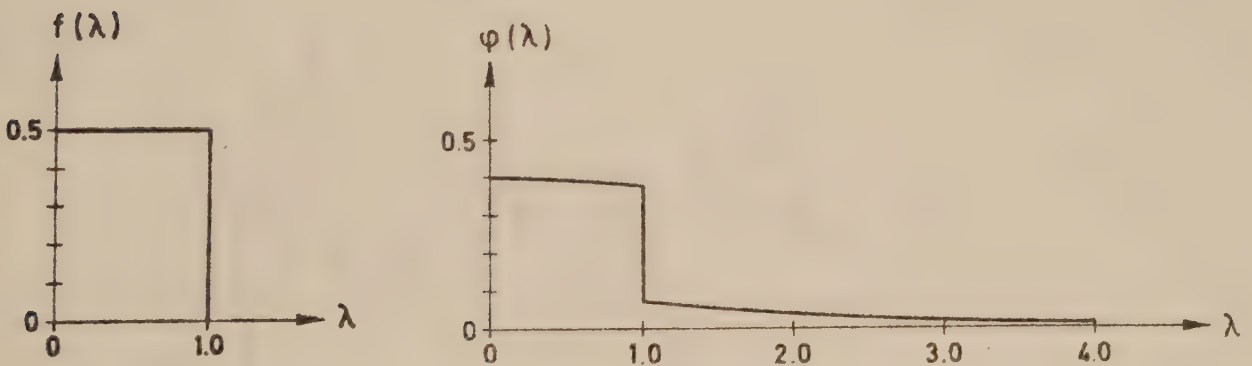


FIGURE 3.3 Spectral density  $f = \frac{1}{2} I_{[-1,1]}$  of  $X$  and spectral density  $\varphi$  of the clipped process  $\xi$ .



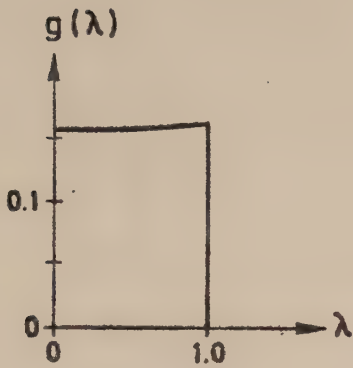


FIGURE 3.4 The function  $g = 2^{-1/2} \pi^{-3/2} f/\varphi$ , where  $f$  and  $\varphi$  are as in Figure 3.3.

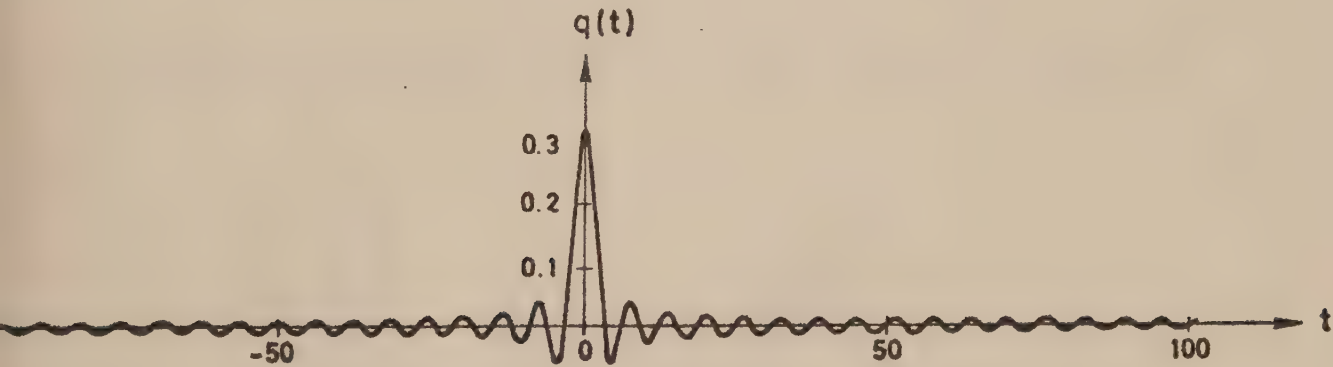


FIGURE 3.5 The impulse response function  $q$ , which is the Fourier transform of  $g$  in Figure 3.4.



FIGURE 3.6 A realization of the Gaussian process  $X$  with spectral density  $f$  as in Figure 3.3.



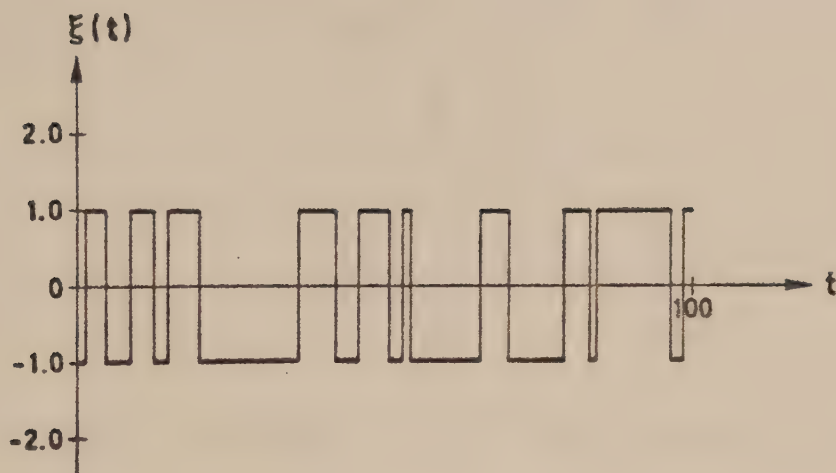


FIGURE 3.7 The clipped process  $\xi = \text{sign } X$  with  $X$  as above.

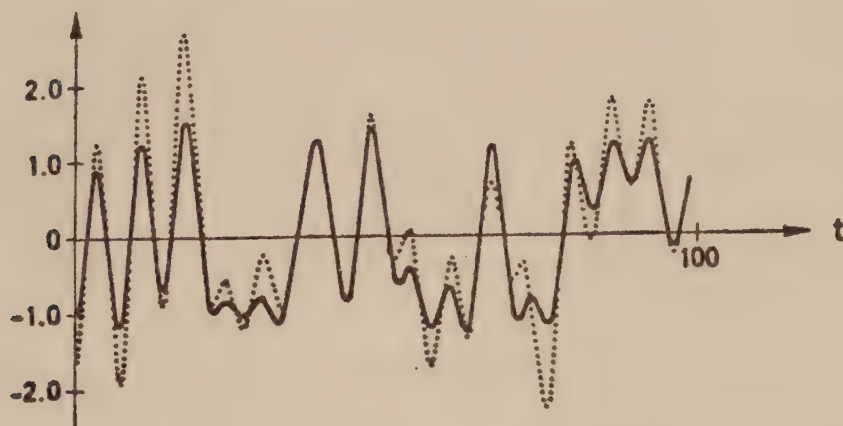


FIGURE 3.8 The interpolator  $\hat{X}$ . The dotted line indicates the actual path of  $X$  in Figure 3.6.

It is reasonable to believe that the white noise case in Example 3.1 is a difficult case for the interpolator. That this may be true is indicated by the more trivial narrow-band case in the following example.

**EXAMPLE 3.2** Here  $X$  is narrow-band Gaussian noise with spectral density specified as  $h^2$  with the function  $h$  as in Figure 3.9. A simulation gave  $\xi$  in Figure 3.10, from which  $\hat{X}$  (Figure 3.11) was calculated.





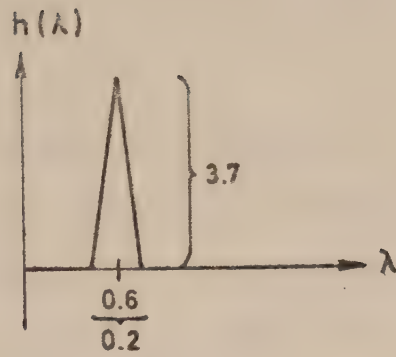


FIGURE 3.9 The function  $h = \sqrt{f}$  in the narrow band case.

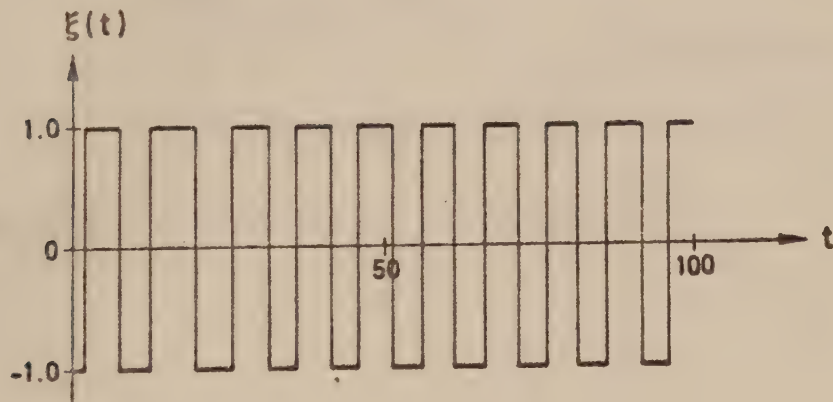


FIGURE 3.10 Clipped narrow band noise.

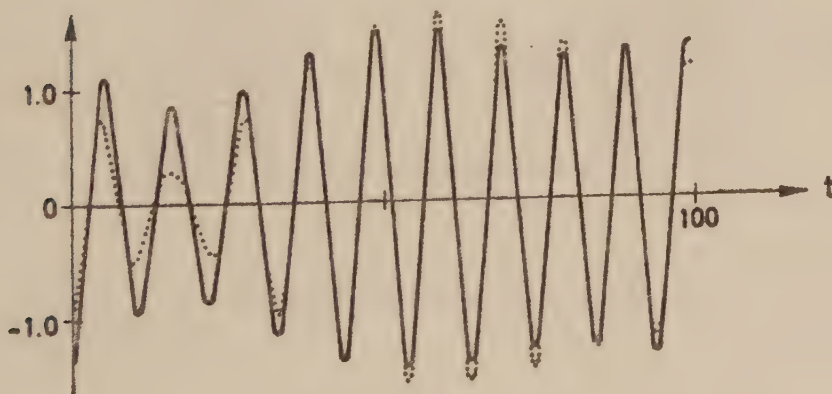


FIGURE 3.11 The interpolator  $\hat{X}$  and the narrow-band noise  $X$  (dotted).

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Another difficult case for the interpolator should be when the sign-changes appear in clusters, so there are comparatively large intervals without zeros.

EXAMPLE 3.3 The last example is low frequency noise with a superposed high frequency component. The spectral density is the square of the function  $h$  in Figure 3.12. The clipped process  $\xi$  in Figure 3.13 gives rise to the realization of  $\hat{X}$  in Figure 3.14.

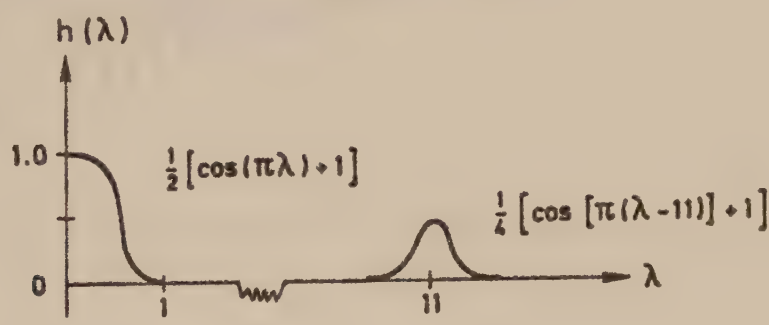


FIGURE 3.12 The function  $h = \sqrt{f}$  in Example 3.3.

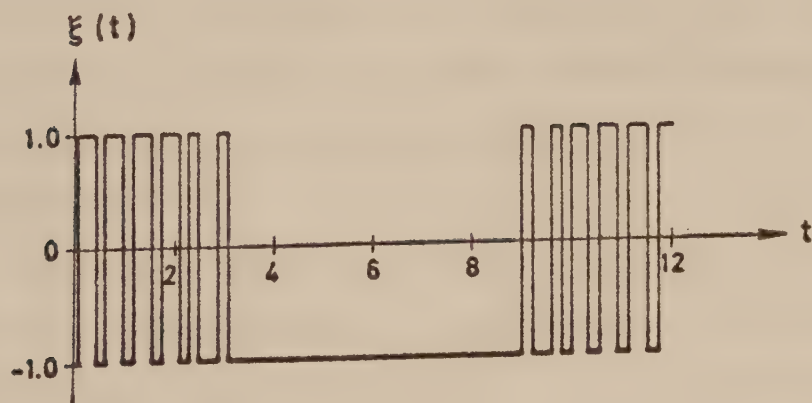


FIGURE 3.13 The sign process in Example 3.3.



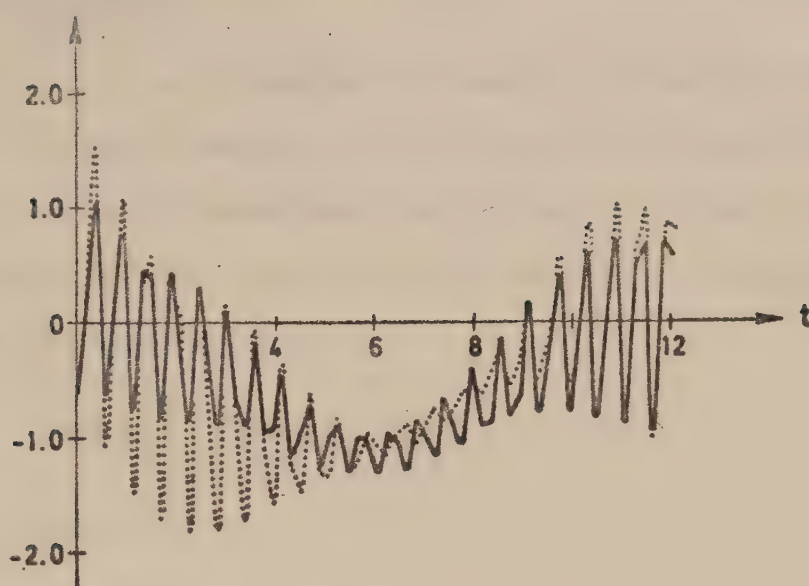


FIGURE 3.14 The reconstruction  $\hat{X}$  and the Gaussian process  $X$  (dotted) of Example 3.3.

We have found that in each of these three examples  $\hat{X}$  follows  $X$  surprisingly well. Most of the error seems to originate from extreme levels of  $X$  which apparently are difficult to find. Since extreme level crossings of  $X$  occur like a Poisson process (Lindgren, de Maré and Rootzén (1975)) it is not surprising that the interpolator has difficulties in detecting them.

It should be noticed that it may very well happen that the interpolator  $\hat{X}$  and the clipped process  $\xi$ , and hence also the original process  $X$ , have different signs. This could seem unsatisfactory but is a consequence of the linearity of our interpolator. It is of course possible to improve on  $\hat{X}$  by correcting its sign, i.e. to define a new interpolator

$$\hat{X}_c(t) = \hat{X}(t) \text{ if } \hat{X}(t) \cdot \xi(t) > 0 \text{ and } 0 \text{ else.}$$

Then  $\hat{X}_c$  is an improvement but according to the simulations the improvement does not seem to be substantial.





#### 4. The interpolation error and other characteristics of the interpolator

In Section 3 we illustrated the behaviour of the best linear interpolator  $\hat{X}$  in some simulations. Here we will derive some analytical properties such as its covariance function, spectral density and the interpolation error. We shall assume that the conclusion in Theorem 1.4 is in force so we can write  $\hat{X}$  as

$$\hat{X}(s) = \int_{-\infty}^{\infty} \xi(s-u) q(u) du.$$

Then the covariance is

$$\begin{aligned} E[\hat{X}(s) \hat{X}(t)] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E[\xi(s-u) \xi(t-v)] q(u) q(v) dudv = \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho(s-t-u+v) q(u) q(v) dudv = \\ &= \int_{-\infty}^{\infty} (\rho * q)(s-t+v) q(v) dv. \end{aligned}$$

Now we have

$$\rho * q = \sqrt{\frac{2}{\pi}} r,$$

by (1.11) and since  $r$  is integrable we get

$$(4.1) \quad E[\hat{X}(s) \hat{X}(t)] = \sqrt{\frac{2}{\pi}} r * q(s-t),$$

which is square integrable. Hence the spectral density of  $\hat{X}$  is

$$2\pi \sqrt{\frac{2}{\pi}} f \cdot g = \frac{2}{\pi} \frac{f^2}{\varphi}.$$

To obtain an analytical expression for the interpolation error

$$R = E\{[X(t) - \hat{X}(t)]^2\},$$

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The Effect of the Diet on the Blood Sugar in Diabetes Mellitus

Wm. C. Calkins, M.D., and J. H. Calkins, M.D.,  
The Effect of the Diet on the Blood Sugar in Diabetes Mellitus

The Effect of the Diet on the Blood Sugar in Diabetes Mellitus

The Effect of the Diet on the Blood Sugar in Diabetes Mellitus

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we use that

$$\hat{X}(t) \perp (X(t) - \hat{X}(t)).$$

Thus

$$E[\hat{X}(t) X(t)] = E[\hat{X}(t) \hat{X}(t)] = \sqrt{\frac{2}{\pi}} r * q(0),$$

by (4.1) and hence

$$\begin{aligned} R &= E\{[X(t) - \hat{X}(t)] X(t)\} \\ &= 1 - \sqrt{\frac{2}{\pi}} r * q(0) = 1 - \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} r(s) q(s) ds. \end{aligned}$$

By Parseval's identity we obtain the familiar expression of  $R$  as

$$R = 1 - 2\pi \int_{-\infty}^{\infty} f(\lambda) g(\lambda) d\lambda = 1 - \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{f^2(\lambda)}{\varphi(\lambda)} d\lambda,$$

(cf. Yaglom (1962) p. 1691.) For the spectra used in the simulation study in section 3 we get

$$R = 0.18, 0.05 \text{ and } 0.20 \text{ respectively.}$$

These results may be compared with the upper bound

$$R \leq E[X(t) - \sqrt{2/\pi} \xi(t)]^2 = 1 - 2/\pi = 0.36.$$

The following theorem summarizes the results in this section,

**THEOREM 4.1** If the Gaussian process  $X$  satisfies the hypothesis of Theorem 1.4 then the best linear interpolator  $\hat{X}$  has the spectral density  $(2/\pi)f^2/\varphi$  and the covariance function  $\sqrt{2/\pi} r*q$  with  $f$ ,  $\varphi$  and  $q$  as in Theorem 1.4. Further the interpolation error  $R = E\{[X(t) - \hat{X}(t)]^2\}$  satisfies

$$\begin{aligned} R &= 1 - \sqrt{2/\pi} \int_{-\infty}^{\infty} r(s) q(s) ds = \\ &= 1 - 2/\pi \int_{-\infty}^{\infty} f^2(\lambda)/\varphi(\lambda) d\lambda \leq 1 - 2/\pi. \end{aligned}$$



## Appendix

To obtain a realization of a process with a given spectral density we used a system developed and implemented by Civ.ing. Per-Ola Börjesson, Lund Institute of Technology (see Börjesson (1974)).

The idea of Börjesson's method is that we have an exact theory for Fourier transformation of  $C^n$  onto  $C^n$ , the so called discrete Fourier transformation.

If one approximates the spectral density by sampling it in  $n$  equidistant points one will get a covariance function as the discrete Fourier transform of the sampled spectral density. This covariance function is the exact transform of the sampled spectral density and is a discrete approximation of the Fourier transform of the original spectral density.

In the same manner one will get a realization of the process as an appropriate transformation of discrete white noise in the frequency domain in analogy with

$$X(t) = \int_{-\infty}^{\infty} \cos(\lambda t) \sqrt{f(\lambda)} dW_1(\lambda) + \int_{-\infty}^{\infty} \sin(\lambda t) \sqrt{f(\lambda)} dW_2(\lambda),$$

where  $W_1$  and  $W_2$  are independent Wiener processes.

The number of sample points  $n$ , used in the simulations is 256 for the spectral densities in Examples 3.1 and 3.2 and 128 for the spectral density in Example 3.3.

The difference in distribution between the continuous time process  $X$  and the discrete time process  $X_D$  actually obtained stems from the fact that  $X_D$  and its related functions are circular and discrete in time and in the frequency domain as well.

In our case to generate white noise we have used a standard routine to get uniformly distributed random numbers  $Y_1, Y_2, \dots$  over  $[0,1]$





which were then transformed to independent Gaussian random numbers

$z_1, z_2, \dots$  by the transformation

$$\begin{cases} z_{2k-1} = \sqrt{-2 \ln y_{2k-1}} \cos(2\pi y_{2k}) \\ z_{2k} = \sqrt{-2 \ln y_{2k-1}} \sin(2\pi y_{2k}). \end{cases}$$

The processes in the simulation study are discrete approximations of the processes which are specified. The simulation study illustrates the approximate behaviour of the interpolator in the continuous case and the exact behaviour in the discrete case.

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Paper [C]

The behaviour of a non-differentiable stationary Gaussian process after a level crossing.





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THE BEHAVIOUR OF A NON-DIFFERENTIABLE  
STATIONARY GAUSSIAN PROCESS  
AFTER A LEVEL CROSSING  
Jacques de Maré

1975:12

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Abstract

A model process is obtained for the behaviour of a non-differentiable but continuous stationary Gaussian process after a level crossing. It is shown that the sampled process conditioned on a crossing of a fixed level in Slepian's sense, converges weakly towards the model process, when the sample distance decreases to zero. Further it is noticed that there is no difference between conditioning on a vertical window and on a horizontal window in this case.

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# THE BEHAVIOUR OF A NON-DIFFERENTIABLE STATIONARY GAUSSIAN PROCESS AFTER A LEVEL CROSSING

by

Jacques de Maré

## 0. INTRODUCTION

Although the asymptotic properties of the stream of high level crossings of Gaussian processes are quite well known under weak conditions on the covariance structure, not much is known about the behaviour of a stationary Gaussian process after a level crossing of moderate height if the process is non-differentiable. In the regular case Slepian (1962) has developed a model process for a stationary Gaussian process conditioned on a level crossing at zero. The method of obtaining model processes has proved useful when calculating crossing interval distributions (Slepian 1962) and amplitude and wave-length distributions in Gaussian noise (Lindgren 1972a, b).

When the Gaussian process is differentiable and ergodic the probabilities calculated in the model process have an ergodic interpretation since e.g.

$$\frac{\#\{\tau: X(\tau+s_1) \leq x_1, \dots, X(\tau+s_n) \leq x_n, X(\tau) = u, \tau \in [0, T]\}}{\#\{\tau: X(\tau) = u \text{ and } \tau \in [0, T]\}}$$

converges with probability one, when  $T$  tends to infinity, and the limit is the probability that the model process is not larger than  $x_1, x_2, \dots, x_n$  at the points  $s_1, s_2, \dots, s_n$  respectively.

On the other hand, if the process is non-differentiable then it has infinitely many crossings of any level strictly between its minimum and maximum in every compact interval and now it is no longer clear what to mean by the ergodic interpretation of a probability

$$P[X(s_1) \leq x_1, \dots, X(s_n) \leq x_n \text{ given } X(0) = u].$$

However, there is another justification for Slepian's description of the behaviour after a level crossing given by Lindgren (1973). Although it is a delicate problem to condition on a level crossing in continuous time,



it is an elementary problem in discrete time, and Lindgren proved that the sampled process converges weakly towards Slepian's model process when the sample distance goes to zero. The sampled process is then considered as a continuous time process by linear interpolation between sample points.

In this paper we will develop a model process for the behaviour of a stationary, continuous and non-differentiable Gaussian process. As a corollary we will get a justification of the model process by proving weak convergence of the linearly interpolated sampled process, conditioned on a level crossing, when the sample distance tends to zero.

## 1. THE MODEL PROCESS

Let  $X$  be a stationary Gaussian process with continuous sample paths, mean zero, unit variance and correlation function  $r$ . If  $X$  is differentiable Slepian (1962) has proved that  $X$  behaves after a crossing of the level  $u$  as the process  $X_u$  with

$$(1.1) \quad X_u(t) = u \cdot r(t) + \Delta \cdot r'(t)/r''(0) + \kappa(t), \quad t \in R$$

where  $\kappa$  is a non-stationary Gaussian process with mean zero and covariance function

$$E[\kappa(s) \cdot \kappa(t)] = r(s-t) - r(s)r(t) + r'(s)r'(t)/r''(0).$$

The stochastic variable  $\Delta$  may be interpreted as the derivative of  $X$  at the level crossing and is distributed independently of  $\kappa$ . The distribution depends on the sense in which  $X$  crosses the level  $u$ . If the conditioning is in horizontal window sense, which has the ergodic interpretation described in the introduction, then  $\Delta$  has a double Rayleigh distribution but if we condition in vertical window sense the distribution is standard normal. When  $X$  is conditioned on an upcrossing at zero instead of a crossing,  $\Delta$  is distributed as Rayleigh and positive normal, respectively.

If  $X$  is a non-differentiable process we will prove that  $X$  behaves after a  $u$ -crossing at zero as





$$(1.2) \quad X_u(t) = u \cdot r(t) + \kappa(t), \quad t \in R,$$

where  $\kappa$  still is a Gaussian process with mean zero but now with the covariance function  $R$ ,

$$R(s,t) = r(s-t) - r(s)r(t).$$

Since there is no term analogous to that containing  $\Delta$  in (1.1), one would expect to get the same representation regardless of the sense in which we condition which also will prove to be the case.

## 2. THE COVARIANCE CONDITIONS

To ensure that the stationary Gaussian process  $X$  with unit variance is non-differentiable we assume that the correlation function  $r$  is not twice differentiable. More specifically we assume that

$$(i) \quad r(t) = 1 - t^\alpha C(t) + o[t^\alpha C(t)], \quad t \downarrow 0, \quad 0 < \alpha < 2,$$

where  $C(\cdot)$  is a slowly varying function at zero. The condition (i) is used by Qualls and Watanabe (1972) to handle high level crossings of non-differentiable Gaussian processes. The conditions we will use, are

$$(ii) \quad r(t) = 1 + o(t^\alpha), \quad t \downarrow 0, \text{ for some } \alpha, \quad 0 < \alpha < 2,$$

and

$$(iii) \quad r(s-t) - r(s+t) = o\{[1-r(2t)]^{1/2}\}, \quad t \downarrow 0 \text{ uniformly in } s,$$

which are consequences of (i). That (ii) follows is immediate, and (i) implies (iii) since, if we choose  $\beta < 1$  with  $\alpha/2 < \beta < \alpha$  then

$$r(s-t) - r(s+t) = o(t^\beta), \quad t \downarrow 0 \text{ uniformly in } s,$$

when (i) holds by Boas (1967, Theorem 1) and hence

$$\frac{r(s-t) - r(s+t)}{[1-r(2t)]^{1/2}} = \frac{o(t^\beta)}{\{(2t)^\alpha C(2t) + o[(2t)^\alpha C(2t)]\}^{1/2}} \rightarrow 0, \quad t \downarrow 0,$$

uniformly in  $s$ , since  $\beta > \alpha/2$ , which establishes (iii).



### 3. THE THEOREM

Here we are going to prove that the non-differentiable Gaussian process  $X$  conditioned on the event

$$(3.1) \quad A_\varepsilon(u) = \{X(-\varepsilon) < u < X(\varepsilon) \text{ or } X(-\varepsilon) > u > X(\varepsilon)\}$$

converges weakly towards the model process  $X_u$  in (1.2) when  $\varepsilon$  tends to zero. We begin by establishing the convergence of the finite-dimensional distributions.

**LEMMA 3.1** Let  $X$  be a stationary Gaussian process with continuous but non-differentiable sample paths, mean zero, unit variance and correlation function satisfying (iii). Then the finite-dimensional distributions of  $X$  conditioned on  $A_\varepsilon(u_0)$  converge to those of the model process  $X_{u_0}$  defined by (1.2).

**PROOF** Define  $U_\varepsilon = [X(\varepsilon) + X(-\varepsilon)]/2$  and  $V_\varepsilon = [X(\varepsilon) - X(-\varepsilon)]/\sigma_\varepsilon$  with  $\sigma_\varepsilon = \sqrt{2[1 - r(2\varepsilon)]}$ . Then

$$A_\varepsilon(u_0) = \{|U_\varepsilon - u_0| \leq \sigma_\varepsilon \cdot |V_\varepsilon|/2\}.$$

We write  $a_\varepsilon(u_0) = \{(u, v) \in R^2: |u - u_0| \leq \sigma_\varepsilon \cdot |v|/2\}$  and use the notation  $\underline{X} \leq \underline{x}$  for  $X(t_1) \leq x_1, \dots, X(t_n) \leq x_n$ . We have

$$P[\underline{X} \leq \underline{x} \mid A_\varepsilon(u_0)] = \frac{\iint_{a_\varepsilon(u_0)} P(\underline{X} \leq \underline{x} \mid U_\varepsilon = u, V_\varepsilon = v) \phi(u/DU_\varepsilon) \phi(v) \, du \, dv}{\iint_{a_\varepsilon(u_0)} \phi(u/DU_\varepsilon) \phi(v) \, du \, dv}.$$

Using the substitution  $u' = (u - u_0)/\sigma_\varepsilon$ , we get the right handside equal to

$$\frac{\iint_{2|u'| \leq |v|} P(\underline{X} \leq \underline{x} \mid U_\varepsilon = u_0 + u'\sigma_\varepsilon, V_\varepsilon = v) \phi\left[\frac{u_0 + u'\sigma_\varepsilon}{DU_\varepsilon}\right] \phi(v) \, du' \, dv}{\iint_{2|u'| \leq |v|} \phi[(u_0 + u')/DU_\varepsilon] \phi(v) \, du' \, dv}.$$

We will prove that the integrands converge pointwise, and hence the integrals converge by dominated convergence, when  $\varepsilon$  tends to zero. The limit will be a multidimensional Gaussian probability expressible by means of the model process  $X_{u_0}$  as



$$\lim_{\varepsilon \downarrow 0} P[\underline{X} \leq \underline{x} \mid A_\varepsilon(u_0)] = P[X_{u_0}(t_1) \leq x_1, \dots, X_{u_0}(t_n) \leq x_n].$$

To prove pointwise convergence consider

$$\begin{aligned} E[X(s) \mid U_\varepsilon = u_0 + u'\sigma_\varepsilon, V_\varepsilon = v] &= (u_0 + u'\sigma_\varepsilon)[r(s+\varepsilon) + r(s-\varepsilon)]/[1+r(2\varepsilon)] + \\ &+ (v/\sigma_\varepsilon)[r(s-\varepsilon) - r(s+\varepsilon)] \rightarrow u_0 r(s), \quad \varepsilon \downarrow 0, \end{aligned}$$

because  $[r(s-\varepsilon) - r(s+\varepsilon)]/\sigma_\varepsilon \rightarrow 0$ ,  $\varepsilon \downarrow 0$ , by (iii). Further the conditional covariance function is given by

$$\begin{aligned} R_\varepsilon(s, t) &= \text{Cov}[X(s), X(t) \mid U_\varepsilon = u_0 + u'\sigma_\varepsilon, V_\varepsilon = v] = \\ &= r(s-t) - \frac{[r(s+\varepsilon)+r(s-\varepsilon)][r(t+\varepsilon)+r(t-\varepsilon)]}{2[1+r(2\varepsilon)]} - \\ &- [r(s-\varepsilon) - r(s+\varepsilon)][r(t-\varepsilon) - r(t+\varepsilon)]/\sigma_\varepsilon^2. \end{aligned}$$

The last term tends to zero by (iii) and hence

$$(3.2) \quad R_\varepsilon(s, t) \rightarrow r(s-t) - r(s)r(t), \quad \varepsilon \downarrow 0,$$

and the pointwise convergence is established.  $\square$

REMARK 3.2 Note that we get the same asymptotic distribution if we condition on the event  $\{X(-\varepsilon) < u < X(\varepsilon)\}$  as if we condition on  $A_\varepsilon(u)$ . Hence when the derivative does not exist, it does not matter if we had an up-crossing or a down-crossing.

THEOREM 3.3 Let  $X$  be a stationary Gaussian process with continuous but non-differentiable sample paths, mean zero and covariance function  $r$  satisfying (ii) and (iii). Then the process  $X$  conditioned on  $A_\varepsilon(u)$  converges weakly towards the model process  $X_u$  of (1.2) in the space of continuous functions with the topology of uniform convergence on compacts.

PROOF The convergence of the finite-dimensional distributions is given by Lemma 3.1. To prove tightness we will check the following sufficient condition (cf Billingsley (1962) p 95).

- (1) The family of conditional distributions of  $X(0)$  given  $A_\varepsilon(u)$ ,  $\varepsilon > 0$  is tight.





$$(2) \quad P[|X(t+h) - X(t)| \geq \lambda \|A_\varepsilon(u)\|] \leq \lambda^{-\xi} |h|^\eta \quad \text{for some } \eta > 1 \quad \text{and } \xi \geq 0.$$

The condition (1) is an immediate consequence of the finite-dimensional distribution convergence. To prove (2) we will use the notations

$\sigma_\varepsilon = \sqrt{2[1-r(2\varepsilon)]}$ ,  $U_\varepsilon = [X(\varepsilon) + X(-\varepsilon)]/2$ ,  $V_\varepsilon = [X(\varepsilon) - X(-\varepsilon)]/\sigma_\varepsilon$  as in the proof of Lemma 3.1. Introduce the conditional standard deviation and mean  $D = \{\text{Var}[X(t+h) - X(t) \mid U_\varepsilon, V_\varepsilon]\}^{1/2}$  and  $M = E[X(t+h) - X(t) \mid U_\varepsilon, V_\varepsilon]$  respectively, and note that  $D$  is a constant. Then

$$\begin{aligned} P &= P[|X(t+h) - X(t)| \geq \lambda \|A_\varepsilon(u)\|] = \\ &= E\{P[|X(t+h) - X(t)| \geq \lambda \|U_\varepsilon, V_\varepsilon\| \mid A_\varepsilon(u)]\} \leq \\ &\leq E\{P[|X(t+h) - X(t)| \geq \lambda \|U_\varepsilon, V_\varepsilon\| \cdot I_{|M|} \leq \lambda/2 \mid A_\varepsilon(u)]\} + \\ &+ P[|M| > \lambda/2 \mid A_\varepsilon(u)]. \end{aligned}$$

The first term is bounded by

$$\begin{aligned} E\{P[|X(t+h) - X(t) - M| \geq \lambda/2 \mid U_\varepsilon, V_\varepsilon] \mid A_\varepsilon(u)\} &= 2[1 - \Phi(\lambda/(2D))] \leq \\ &\leq \sqrt{2/\pi} 2D/\lambda e^{-\lambda^2/(8D^2)}. \end{aligned}$$

To estimate the second term, note that

$$M = U_\varepsilon \cdot a + V_\varepsilon \cdot b$$

with  $a = [r(t+h+\varepsilon) + r(t+h-\varepsilon) - r(t+\varepsilon) - r(t-\varepsilon)]/[1+r(2\varepsilon)]$  and

$b = [r(t+h+\varepsilon) - r(t+h-\varepsilon) - r(t+\varepsilon) + r(t-\varepsilon)]/\sigma_\varepsilon$ . Since the event  $A_\varepsilon(u)$  is equivalent to  $\{|U_\varepsilon - u| \leq \sigma_\varepsilon \cdot |V_\varepsilon|/2\}$ , we get that, conditional on  $A_\varepsilon(u)$ ,

$$|M| \leq |V_\varepsilon| (|a|\sigma_\varepsilon/2 + |b|) + |u \cdot a| = |V_\varepsilon| \cdot c + |u \cdot a|.$$

For all sufficiently small  $h$  we have  $|u \cdot a| < \lambda/4$  and then

$$P[|M| > \lambda/2 \mid A_\varepsilon(u)] \leq P[|V_\varepsilon| \cdot c > \lambda/4 \mid A_\varepsilon(u)] < B \cdot e^{-\lambda^2/(32c^2)}$$

for some sufficiently large constant  $B$ .

Now we have to look closer at the numbers  $D$  and  $c$  which appear in the upper bounds. Computing  $D$ ,

$$D^2 = 2[1-r(h)] - a^2[1+r(2\varepsilon)] - b^2 \leq 2[1-r(h)] = o(h^\alpha)$$



by (ii) if  $\varepsilon$  is not too large. Hence  $a = o(h^{\alpha/2})$  and  $b = o(h^{\alpha/2})$  and then  $c$  is of the same order. Finally we get

$$P = \lambda^{-1} o(h^{\alpha/2}) e^{-\lambda^2/o(h^\alpha)} + B e^{-\lambda^2/o(h^\alpha)} = o(h^2),$$

which establishes the tightness.

COROLLARY 3.4 Let  $X_\varepsilon$  be the continuous process which is obtained by sampling  $X$  at time points  $(2k+1)\varepsilon$ ,  $k = 0, \pm 1, \pm 2, \dots$ , and interpolating linearly in between. Then  $X_\varepsilon$  conditioned on  $A_\varepsilon(u)$  in (3.1) converges weakly towards  $X_u$  defined by (1.2).

PROOF For all  $T > 0$

$$P\left[\sup_{-T \leq s \leq T} |X_\varepsilon(s) - X(s)| > \delta \mid A_\varepsilon(u)\right] \leq P\left[\sup_{\substack{-T \leq s \leq T \\ |h| \leq 2\varepsilon}} |X(s+h) - X(s)| > \delta \mid A_\varepsilon(u)\right].$$

But for all  $\eta > 0$  there are  $\varepsilon_1$  and  $\varepsilon_2$  such that for all  $\varepsilon' < \varepsilon_1$  and  $\varepsilon'' < \varepsilon_2$

$$P\left[\sup_{\substack{-T \leq s \leq T \\ |h| < \varepsilon'}} |X(s+h) - X(s)| > \delta \mid A_{\varepsilon''}(u)\right] < \eta$$

(cf Billingsley (1968) p 55). Hence if  $\varepsilon$  is smaller than both  $\varepsilon_1$  and  $\varepsilon_2$ ,

$$P\left[\sup_{-T \leq s \leq T} |X_\varepsilon(s) - X(s)| > \delta \mid A_\varepsilon(u)\right] < \eta,$$

i.e.  $X_\varepsilon(\cdot) - X(\cdot)$  conditioned on  $A_\varepsilon(u)$  tends to zero in probability in the function space and since  $X(\cdot)$  conditioned on  $A_\varepsilon(u)$  converges weakly, so does  $X_\varepsilon(\cdot)$ .

The next corollary gives an ergodic interpretation of the model process  $X_u$  of (1.2).

COROLLARY 3.5 Let  $X$  be ergodic and  $X_\varepsilon$  as in Corollary 3.4. Then

$$\begin{aligned} P[X_u(t_1) \leq x_1, \dots, X_u(t_n) \leq x_n] = \\ = \lim_{\varepsilon \downarrow 0} \lim_{T \rightarrow \infty} \frac{\#\{\tau \in \Gamma_T: X_\varepsilon(\tau+t_1) \leq x_1, \dots, X_\varepsilon(\tau+t_n) \leq x_n, X_\varepsilon(\tau-\varepsilon) < u, X_\varepsilon(\tau+\varepsilon) > u\}}{\#\{\tau \in \Gamma_T: X_\varepsilon(\tau-\varepsilon) < u, X_\varepsilon(\tau+\varepsilon) > u\}} \end{aligned}$$

with probability one,



where  $\Gamma_T = \{\tau: |\tau| \leq T \text{ and } \tau = 0, \pm 2\varepsilon, \pm 4\varepsilon, \dots\}$ .

PROOF The inner limit converges to a probability by ergodicity when  $T$  tends to infinity. This probability tends to the left handside when  $\varepsilon$  goes to zero by Corollary 3.4.

COROLLARY 3.6 When  $X$  is non-differentiable, conditioning in horizontal window sense and in vertical-window sense in the sampled process give asymptotically the same result, when the sample distance tends to zero.

REMARK 3.7 To prove the asymptotic Poisson property of high level crossings of a Gaussian process with correlation function satisfying (i), Pickands (1969) and Qualls and Watanabe (1972) introduced a non-stationary Gaussian process  $Y$  with mean  $E[Y(t)] = -|t|^\alpha$  and covariance  $\text{Cov}[Y(s), Y(t)] = |s|^\alpha + |t|^\alpha - |s-t|^\alpha$ . The finite-dimensional distributions of a suitably normed version  $Z_u$  of the model process  $X_u$  in (1.2) converge towards those of  $Y$ , when the level  $u$  tends to infinity. In the light of Remark 3.2 and Corollary 3.6 this fact is hardly surprising. Let  $G(\cdot)$  denote the inverse of  $\sqrt{|\cdot|^\alpha C(\cdot)}$  (cf Qualls and Watanabe 1972 p 582) and  $Z_u(t) = u\{X_u[tG(1/u)] - u\}$ . Then we get

$$\begin{aligned} E[Z_u(t)] &= u^2\{r[tG(1/u)] - 1\} = \\ &= u^2[-|tG(1/u)|^\alpha C[tG(1/u)] - o\{G^\alpha(1/u)C[tG(1/u)]\}] = \\ &= -u^2|t|^\alpha/u^2 + o(1) \rightarrow -|t|^\alpha, \quad u \rightarrow \infty, \end{aligned}$$

because of the definition of  $G$  and the fact that  $C(\cdot)$  is slowly varying.

In the same way

$$\begin{aligned} \text{Cov}[Z_u(s), Z_u(t)] &= u^2\{r[(s-t)G(1/u)] - r[sG(1/u)]r[tG(1/u)]\} \rightarrow \\ &\rightarrow |s|^\alpha + |t|^\alpha - |s-t|^\alpha, \quad u \rightarrow \infty. \end{aligned}$$





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